# Darboux-Bäcklund Derivation of Rational Solutions of the Painlevé IV Equation

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## ABSTRACT

Rational solutions of the Painlevé IV equation are constructed in the setting of pseudo-differential Lax formalism describing AKNS hierarchy subject to the additional non-isospectral Virasoro symmetry constraint. Convenient Wronskian representations for rational solutions are obtained by successive actions of the Darboux-Bäcklund transformations.

## 1   Introduction

Two main observations are put together in this paper with the purpose of developing a systematic method rooted in Darboux-Bäcklund techniques within the pseudo-differential Lax formalism to generate rational solutions of the Painlevé IV equation. First, we observe that the AKNS hierarchy subject to the additional non-isospectral Virasoro symmetry constraint (also known as the string equation) reduces to the Painlevé IV equation. Secondly, we make use of the fact that the Darboux-Bäcklund (DB) transformations commute with the additional-symmetry Virasoro flows [2] and thus these transformations can effectively be used to construct all known rational solutions [5, 6, 10, 12] of the Painlevé IV equation (4.3) out of a few basic ones.

## 2   AKNS hierarchy with string condition

To introduce the AKNS Lax formalism with additional Virasoro symmetry flows we follow [2] and define the pseudo-differential Lax hierarchy as described by a dressing

formula
$$L = W \partial_x W^{-1} = \partial_x - r \partial_x^{-1} q, \quad W = 1 + \sum_1^\infty w_n \partial_x^{-n} \qquad (2.1)$$

The associated $t_n$-flows of this hierarchy:

$$\frac{\partial r}{\partial t_n} = L_+^n(r) \qquad ; \qquad \frac{\partial q}{\partial t_n} = -\left(L^*\right)_+^n(q) \qquad n = 1, 2, \ldots \qquad (2.2)$$

reproduce for $n = 2$ the AKNS equations :

$$\frac{\partial}{\partial t_2} q + q_{xx} - 2q^2 r = 0, \qquad \frac{\partial}{\partial t_2} r - r_{xx} + 2qr^2 = 0 \,. \qquad (2.3)$$

Furthermore, it is useful to introduce as in [3] a squared eigenfunction $\rho$ such that

$$\rho_x = -2rq$$

and, which in particular, satisfies [3] :

$$\frac{\partial}{\partial t_2} \rho = 2\left(-qr_x + rq_x\right) \,. \qquad (2.4)$$

The above formalism is augmented by the Orlov-Schulman operator [14] :

$$M = W \left( \sum_{l \geq 1} l t_l \partial_x^{l-1} \right) W^{-1}$$

conjugated to the Lax operator according to the commutation relation $[L, M] = 1$.

The additional Virasoro symmetry flows are then defined as

$$\bar{\partial}_{k,n} L = -\left[\left(M^n L^k\right)_-, L\right] = \left[\left(M^n L^k\right)_+, L\right] + n M^{n-1} L^k \qquad (2.5)$$

and they commute with the usual isospectral flows as defined by $\partial L / \partial t_n = [L_+^n, L]$ or (2.2).

For $n = 1$ formula (2.5) yields

$$\left(\bar{\partial}_{k,1} L\right)_- = \left[\left(M L^k\right)_+, L\right]_- + \left(L^k\right)_- \,, \qquad (2.6)$$

which can be rewritten as

$$\left(\bar{\partial}_{k,1} L\right)_- = \left(M L^k\right)_+ (r) \partial^{-1} q - r \partial^{-1} \left(M L^k\right)_+^* (q) + \sum_{j=0}^{k-1} L^{k-j-1}(r) \partial^{-1} \left(L^*\right)^j (q) \,. \qquad (2.7)$$

For $k = 0, 1, 2$ these flows form the $sl(2)$ subalgebra of the Virasoro algebra and preserve the form of the Lax operator of the AKNS hierarchy. Subsequently, they can equivalently be described by their action on the eigenfunction $r$ and the adjoint eigenfunction $q$ :

$$\left(\partial_\tau L\right)_- = \left(\partial_\tau r\right) \partial^{-1} q + r \partial^{-1} \left(\partial_\tau q\right) \qquad (2.8)$$

with $\partial_\tau \equiv \bar\partial_{k,1}$ $(k = 0, 1, 2)$. Explicitly, one finds [2] :

$$\bar\partial_{0,1} r = (M)_+ (r), \qquad\qquad \bar\partial_{0,1} q = - (M)_+^* (q), \qquad\qquad (2.9)$$
$$\bar\partial_{1,1} r = (ML)_+ (r) + (1 - \nu)r, \qquad \bar\partial_{1,1} q = - (ML)_+^* (q) + \nu q, \qquad (2.10)$$
$$\bar\partial_{2,1} r = \left(ML^2\right)_+ (r) + L(r), \qquad \bar\partial_{2,1} q = - \left(ML^2\right)_+^* (q) + L^*(q). \qquad (2.11)$$

Note arbitrariness in equation (2.10) expressed by a parameter $\nu$.

In what follows we impose the symmetry conditions:

$$\bar\partial_{1,1} r = 0, \quad \bar\partial_{1,1} q = 0\,. \qquad\qquad (2.12)$$

Since

$$ML = W \left(\sum_{l \geq 1} l t_l \partial_x^l\right) W^{-1}, \quad (ML)^* = W^{-1\,*} \left(-\partial_x x + \partial_x^2 2t_2 + \ldots\right) W^*$$
$$= -1 + W^{-1\,*} \left(-x\partial_x + 2t_2 \partial_x^2 + \ldots\right) W^*$$

it follows that

$$(ML)_+ = x\partial_x + 2t_2 (L)_+^2 + \ldots, \quad - (ML)_+^* = 1 + x\partial_x - 2t_2 (L^*)_+^2 + \ldots$$

Thus in view of equation (2.10) the condition (2.12) amounts to

$$-xq_x - 2t_2 \frac{\partial}{\partial t_2} q = q + \nu q, \quad xr_x + 2t_2 \frac{\partial}{\partial t_2} r = -r + \nu r$$
$$x\rho_x + 2t_2 \frac{\partial}{\partial t_2} \rho = -\rho\,. \qquad\qquad (2.13)$$

The model we consider is obtained by applying the string equation (2.13) to eliminate $t_2$-dependence from $r, q, \rho$ and setting the higher flows to zero. Setting $t_2 = -1/4$ and eliminating $t_2$-flow dependence from eq. (2.13) by inserting values from (2.3)-(2.4) yields :

$$-xq_x + \frac{1}{2} \left(-q_{xx} + 2q^2 r\right) = q + \nu q, \quad xr_x - \frac{1}{2} \left(r_{xx} - 2qr^2\right) = -r + \nu r \qquad (2.14)$$
$$\rho + x\rho_x = \rho - 2xrq = q_x r - qr_x$$

Multiplying the first equation by $r_x$ and second by $q_x$ and adding them one obtains:

$$\left(q_x r_x - q^2 r^2\right)_x = 2(q_x r - qr_x) - 2\nu(qr)_x$$

or after integration :

$$q_x r_x = q^2 r^2 + 2x\rho - 2\nu rq - \left(\mu^2 - \nu^2\right), \qquad\qquad (2.15)$$

where we have set the integration constant to $\mu^2 - \nu^2$, with $\mu$ being an extra parameter. In what follows $\mu$ will, in addition to $\nu$, parametrize solutions of the Painlevé IV

equation. The origin of $\mu$ is very transparent in an algebraic approach to integrable models subject to the scaling condition [4].

Dividing the first of eqs. (2.14) by $q$ and second by $r$ and summing them yields:

$$2\nu = x\left(-\frac{q_x}{q} + \frac{r_x}{r}\right) - \frac{1}{2}\left(\frac{(rq)_{xx}}{rq} - 2\frac{r_x q_x}{rq}\right) + 2qr =$$

$$= x\left(2x - \frac{\rho}{rq}\right) - \frac{1}{2}\left(\frac{(rq)_{xx}}{rq} - 2\frac{r_x q_x}{rq}\right) + 2qr,$$

where in the last line we used the third of eqs. (2.14). Inserting $q_x r_x$ from eq. (2.15) produces after multiplication by $rq$ :

$$2x^2 rq + x\rho - \frac{1}{2}(rq)_{xx} - 4\nu rq + 3(rq)^2 = \mu^2 - \nu^2 .$$

Inserting in the above $rq = -\rho_x/2$ results in an equation entirely expressed in terms of only one variable $\rho$ :

$$-x^2\rho_x + x\rho + \frac{1}{4}\rho_{xxx} + 2\nu\rho_x + \frac{3}{4}\rho_x^2 = \mu^2 - \nu^2 . \tag{2.16}$$

One recognizes in the above equation the special case of the Chazy I equation [7]. After multiplication of eq. (2.16) by $\rho_{xx}$ we can rewrite the resulting equation as a total derivative, which after an integration becomes

$$\rho_{xx}^2 = 4\left(x\rho_x - \rho\right)^2 - 2\rho_x^3 - 8\nu\rho_x^2 + 8(\mu^2 - \nu^2)\rho_x - 8C, \tag{2.17}$$

with $C$ being an unknown integration constant. Another well-known form of this equation is [9] :

$$\rho_{xx}^2 = 4\left(x\rho_x - \rho\right)^2 - 2\prod_{i=1}^{3}(\rho_x + v_i) .$$

By comparison with eq. (2.17) one obtains

$$v_1 + v_2 + v_3 = 4\nu, \quad v_1 v_2 + v_1 v_3 + v_2 v_3 = -4(\mu^2 - \nu^2), \quad v_1 v_2 v_3 = 4C.$$

If we set the integration constant $C$ to zero then eq. (2.17) can be simplified as follows :

$$\rho_{xx}^2 = 4\left(x\rho_x - \rho\right)^2 - 2\rho_x\left[\rho_x - 2(\mu - \nu)\right]\left[\rho_x + 2(\mu + \nu)\right] \tag{2.18}$$

or in a slightly different and more convenient form :

$$\left(2\left(x\rho_x - \rho\right) + \rho_{xx}\right)\left(2\left(x\rho_x - \rho\right) - \rho_{xx}\right) = 2\rho_x\left[\rho_x - 2(\mu - \nu)\right]\left[\rho_x + 2(\mu + \nu)\right] . \tag{2.19}$$

First, one notes that equation (2.19) is manifestly invariant under $\mu \to -\mu$ and thus a solution of equation (2.19) with one value of $\mu$ solves this equation for $-\mu$ as well. Next, one notes that the left hand side of equation (2.19) remains invariant under substitution $\rho = \widetilde{\rho} + kx$ with $k$ being a constant. For $k = 2(\mu - \nu)$ and $k = -2(\mu + \nu)$ the right hand side can be given the form $2\widetilde{\rho}_x[\widetilde{\rho}_x - 2(\widetilde{\mu} - \widetilde{\nu})][\widetilde{\rho}_x + 2(\widetilde{\mu} + \widetilde{\nu})]$ with $\widetilde{\nu} = 3\mu/2 - \nu/2, \widetilde{\mu} = \pm(\mu + \nu)/2$ and $\widetilde{\nu} = -3\mu/2 - \nu/2, \widetilde{\mu} = \pm(\mu - \nu)/2$, respectively. Hence for these values of $k$ the transformation $\rho \to \widetilde{\rho}$ takes the "old" solution of equation (2.19) to the "new" solution of equation (2.19) with the new parameters $\widetilde{\mu}, \widetilde{\nu}$.

# 3   Basic Polynomial Solutions

One finds by inspection that there exists a class of polynomial solutions of equation (2.19) for $\rho$ being of the form $\rho = bx^3 + cx$. With both coefficients, $b$ and $c$, being non-zero the polynomial solution is :

$$\rho = \frac{8}{27}x^3 \pm \frac{4}{3}x, \quad \mu^2 = \frac{1}{9}, \quad \nu = \mp 1. \tag{3.1}$$

For $c = 0$, one finds that the unique non-zero solution requires $b = 8/27$ with $\nu = 0, \mu^2 = 4/9$ :

$$\rho = \frac{8}{27}x^3, \quad \mu^2 = \frac{4}{9}, \quad \nu = 0. \tag{3.2}$$

Finally, one notes that setting $b = 0$ causes both $x\rho_x - \rho$ and $\rho_{xx}$ to vanish together with the left hand side of eq. (2.19). Thus $\rho = cx$ is a solution to eq. (2.19) for three values of $c$, namely $c = 0, 2(\mu - \nu), -2(\mu + \nu)$ for which the right hand side of eq. (2.19) vanishes as well.

In a text below, the polynomial solutions of equation (2.19) will serve as seeds of chains of the Darboux-Bäcklund transformations.

For $\rho = cx$ the product $rq$ is a constant, $-c/2$. In addition, $q_x r - q r_x = \rho - 2xrq = 2cx$ and thus $rq_x = cx$, $r_x q = -cx$. Multiplying eq. (2.15) by $r$ and then by $q$ one gets from $rq_x = cx$, $r_x q = -cx$ that

$$cxr_x = \left(2cx^2 + c^2/4 + c\nu - (\mu^2 - \nu^2)\right)r, \quad -cxq_x = \left(2cx^2 + c^2/4 + c\nu - (\mu^2 - \nu^2)\right)q$$

For $c = 2(\mu - \nu), -2(\mu + \nu)$ the above equations simplify to

$$xr_x = 2x^2 r, \quad -xq_x = 2x^2 q \ , \quad \rightarrow \quad r = k_1 e^{x^2}, \quad q = k_{-1} e^{-x^2}$$

and as long $c = -2k_1 k_{-1} \neq 0$ the above solutions do not lead to non-zero solutions of the Painlevé IV equation.

We now consider the case of $b = c = 0$. Then $rq = 0$ with either $r = 0$ or $q = 0$ and it follows from (2.15) that $\mu^2 = \nu^2$.

Another useful expression for eqs. (2.14) is :

$$\left(e^{-x^2} r_x\right)_x + 2(\nu - 1)e^{-x^2} r = 2(rq)e^{-x^2} r, \quad \left(e^{x^2} q_x\right)_x + 2(\nu + 1)e^{x^2} q = 2(rq)e^{x^2} q \tag{3.3}$$

For $q = 0$ and $\nu - 1 = n$ being a positive (or zero) integer the first equation of eqs. (3.3) is Hermite's equation, alternatively known as $r_{xx} - 2xr_x + 2nr = 0$ with $r(x) = H_{\nu-1}(x)$ being a solution. Thus one obtains, $r = H_0 = 1$ for $\nu = 1$, $r = H_1 = 2x$ for $\nu = 2$ (the solution we have already studied) and $r = H_2 = 4x^2 - 2$ for $\nu = 3$ etc.

For $-\nu \geq 0$ being a positive integer a substitution

$$r(x) = e^{x^2} g(x)$$

leads to the following equation

$$g_{xx} + 2xg_x + 2\nu g = g_{xx} + 2xg_x - 2mg = 0, \quad m = -\nu = 0, 1, 2, \ldots$$

which has a polynomial solution

$$g_m(x) = \widehat{H}_m(x) = e^{-x^2}\frac{d^m}{dx^m}e^{x^2} = (-\mathrm{i})^m H_m(\mathrm{i}\,x) \tag{3.4}$$

with $g_{m=0} = 1$, $g_{m=1} = 2x$, $g_{m=2} = 2(1 + 2x^2)$ and $g_{m=3} = (3 + 2x^2)4x$ etc. To summarize :

$$r(x) = \begin{cases} H_{\nu-1}, & \text{if } \nu > 0 \text{ is integer} \\ e^{x^2}\widehat{H}_{-\nu}(x), & \text{if } \nu \leq 0 \text{ is integer} \end{cases} \tag{3.5}$$

for $q = 0$. Substituting

$$q = e^{-x^2}f(x),$$

into (3.3) with $r = 0$ one gets the following equation for $f$ :

$$f_{xx} - 2xf_x + 2\nu f = 0,$$

which is a Hermite's equation for $H_n(x)$ for $\nu = n \geq 0$. Thus $f(x) = H_\nu(x) = H_n(x)$ with $n = 0, 1, 2, \ldots$ and $q(x) = \exp(-x^2)H_\nu(x)$ as long as $\nu$ is a positive integer. For $\nu + 1 = -m$, where $m = 0, 1, 2, \ldots$ the equation for $q$,

$$q_{xx} + 2xq_x - 2mq = 0$$

is identical to equation for $g$ written above and has the same solutions. To summarize:

$$q(x) = \begin{cases} e^{-x^2}H_\nu, & \text{if } \nu \geq 0 \text{ is integer} \\ \widehat{H}_{-\nu-1}(x), & \text{if } \nu < 0 \text{ is integer} \end{cases} \tag{3.6}$$

for $r = 0$.

# 4    Painlevé IV equation and its Hamiltonian formalism

In terms of variables:

$$y = -\frac{q_x}{q} - 2x, \quad w = \frac{r_x}{r} - 2x, \quad z = -2rq + 2(\mu + \nu) \tag{4.1}$$

we can express equations (2.14) as

$$y_x = z + y^2 + 2xy - 2\mu, \qquad w_x = -z - w^2 - 2xw + 2\mu$$

$$z_x = 2rq(y - w) = -\frac{z^2}{2w} + wz - 2(\mu + \nu)w + 2\mu\frac{z}{w} \tag{4.2}$$

$$= \frac{z^2}{2y} - yz + 2(\mu + \nu)y - 2\mu\frac{z}{y}$$

Applying one more derivative on the first equation and eliminating $z$ yields the Painlevé IV equation:

$$y_{xx} = \frac{1}{2y}y_x^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 + \nu + 1)y - 2\frac{\mu^2}{y} \tag{4.3}$$

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Repeating the same procedure for the equation with $w$ yields almost identical equation (note however a shift in $\nu$ as compared to eq. (4.3)):

$$w_{xx} = \frac{1}{2w}w_x^2 + \frac{3}{2}w^3 + 4xw^2 + 2(x^2 + \nu - 1)w - 2\frac{\mu^2}{w} \tag{4.4}$$

As found by Okamoto [13], equations (4.2) for a pair of variables $(y, z)$ possess a Hamiltonian representation. Define namely

$$H = -2P^2Q + \left(Q^2 + 2xQ - 2\mu\right)P + \frac{1}{2}(\mu + \nu)Q. \tag{4.5}$$

Then the Hamilton equations are:

$$Q_x = \frac{\partial H}{\partial P} = -4QP + Q^2 + 2xQ - 2\mu \tag{4.6}$$

$$P_x = -\frac{\partial H}{\partial Q} = 2P^2 - 2QP - 2xP - \frac{1}{2}(\mu + \nu). \tag{4.7}$$

They agree with eqs. (4.2) for $Q = y$ and $z = -4QP$ or $P = -z/4y$. Due to the Hamilton equations it holds that

$$\frac{d}{dx}H = H_x = 2QP = -\frac{1}{2}z = rq - (\nu + \mu) = -\frac{1}{2}\rho_x - (\nu + \mu). \tag{4.8}$$

Thus, up to a constant,

$$H = -\frac{1}{2}\rho - x(\nu + \mu). \tag{4.9}$$

Plugging the product $QP = -\left(\rho_x + 2(\mu + \nu)\right)/4$ into expression (4.5) for $H$ and using eq. (4.9) one gets

$$P(-rq - \mu + \nu) + Q\frac{1}{2}rq + xrq = -\frac{1}{2}\rho$$

or

$$\frac{1}{2}P(\rho_x - 2(\mu - \nu)) - \frac{1}{4}Q\rho_x = \frac{1}{2}(x\rho_x - \rho).$$

Define:

$$P = P_0\frac{2}{\rho_x - 2(\mu - \nu)}, \qquad Q = -4Q_0\frac{1}{\rho_x}. \tag{4.10}$$

Then

$$P_0 + Q_0 = \frac{1}{2}(x\rho_x - \rho)$$

$$P_0Q_0 = \frac{1}{32}\rho_x(\rho_x - 2(\mu - \nu))(\rho_x + 2(\mu + \nu)) = \frac{1}{64}\left(4(x\rho_x - \rho)^2 - \rho_{xx}^2\right)$$

$$= \frac{1}{64}(2(x\rho_x - \rho) + \rho_{xx})(2(x\rho_x - \rho) - \rho_{xx})$$

where use was made of eq. (2.18). Solutions to the above equation are easily found to be

$$P_0 = \frac{1}{8}(2(x\rho_x - \rho) \pm \rho_{xx}), \quad Q_0 = \frac{1}{8}(2(x\rho_x - \rho) \mp \rho_{xx}),$$

which gives two answers for expression for the Painlevé IV solution $y = Q$ in terms of the solution to the $\rho$-equation (2.19)

$$y_\pm = Q = \frac{-1}{2\rho_x} \left( 2 \left( x\rho_x - \rho \right) \pm \rho_{xx} \right) \tag{4.11}$$

The above ambiguity in signs can be explained by comparing with the third equation in (4.2). Consider namely the difference $y_+ - y_-$ of two solutions from eq. (4.11):

$$y_+ - y_- = -\frac{\rho_{xx}}{\rho_x} = \frac{z_x}{2rq} = y - w$$

where we used the third of equations in (4.2). Thus $y_+ = y$, $y_- = w$. It is natural to connect these associations by a Bäcklund transformation by defining a "barred" system such that

$$w = y_- = \bar{y}_+ = \bar{y} = \frac{-1}{2\bar{\rho}_x} \left( 2 \left( x\bar{\rho}_x - \bar{\rho} \right) + \bar{\rho}_{xx} \right)$$

and thus the new $\bar{\rho}$ variable needs to satisfy

$$\frac{\rho}{\rho_x} + \frac{\rho_{xx}}{2\rho_x} = \frac{\bar{\rho}}{\bar{\rho}_x} - \frac{\bar{\rho}_{xx}}{2\bar{\rho}_x} \, .$$

From the last of equations (2.14) one can obtain expressions for the right and the left hand side of the above relation in terms of $\bar{q}$ and $r$. This results in a condition :

$$(\ln r)_x = -(\ln \bar{q})_x, \quad \rightarrow \quad \bar{q} = \frac{K}{r}, \quad K = \text{const} \,, \tag{4.12}$$

which also easily follows from the equality $w = \bar{y}$. The above relation resembles an effect of the Darboux-Bäcklund transformation to be described in the next section. Note that the above transformation maps $w$ to $\bar{y}$ and therefore comparing with values of $\nu$ in eqs. (4.3) and (4.4) we conclude that it lowers $\nu$ by 2.

For completeness, let us mention that as follows from eqs. (4.7) for $Q = y$ and $P = -z/4y$, the function $Y = -2P = z/2y$ also satisfies the Painlevé IV equation (4.3) with the parameters $\widetilde{\nu} = 3\mu/2 - \nu/2 - 2$, $\widetilde{\mu} = \pm(\mu + \nu)/2$. By comparing with definitions (4.10) and definitions of $P_0, Q_0$ we find that the transformation $y \rightarrow Y$ is a superposition of the transformation $\rho \rightarrow \widetilde{\rho}$ discussed below equation (2.19) with the Darboux-Bäcklund transformation $y \rightarrow w$ lowering $\nu$ by 2. In what follows we will only discuss the latter transformation together with its square-root.

# 5   Darboux-Bäcklund transformations

The Darboux-Bäcklund transformation is here introduced as a similarity transformation with an operator $T = r\partial_x r^{-1}$ acting on the pseudo-differential AKNS Lax operator through :

$$L = \partial_x - r\partial_x^{-1}q \quad \rightarrow \quad \bar{L} = T \, L \, T^{-1} = \partial_x - \bar{r}\partial_x^{-1}\bar{q}$$

A simple calculation yields

$$\bar{r} = r(\ln r)_{xx} - r^2 q, \quad \bar{q} = -\frac{1}{r}. \tag{5.1}$$

in agreement with relation (4.12) for $K = -1$. Taking a product of $\bar{r}, \bar{q}$ one obtains

$$\bar{\rho}_x = -2\bar{r}\bar{q} = \rho_x + 2(\ln r)_{xx}.$$

Thus the DB transformation (5.1) yields

$$\bar{\rho} = \rho + 2(\ln r)_x = \rho + 2w + 4x = \rho + 2y_- + 4x$$
$$= \rho + 2\frac{-1}{2\rho_x}\left(2\left(x\rho_x - \rho\right) - \rho_{xx}\right) + 4x = \rho + 2\frac{\rho}{\rho_x} + \frac{\rho_{xx}}{\rho_x} + 2x \tag{5.2}$$

or including a reference to the $(\mu, \nu)$ parameters

$$\rho^{(\mu,\nu-2)} = \rho^{(\mu,\nu)} + 2\frac{\rho^{(\mu,\nu)}}{\rho_x^{(\mu,\nu)}} + \frac{\rho_{xx}^{(\mu,\nu)}}{\rho_x^{(\mu,\nu)}} + 2x. \tag{5.3}$$

It is equally easy to formulate the adjoint Darboux-Bäcklund transformation which increases $\nu$ by $+2$. The adjoint Darboux-Bäcklund transformation involves acting with $S = q\partial_x q^{-1}$ on the pseudo-differential Lax operator through :

$$L = \partial_x - r\partial_x^{-1}q \quad \rightarrow \quad \widetilde{L} = S^{*\,-1}\,L\,S^* = (q^{-1}\partial_x^{-1}q)L(q^{-1}\partial_x q) = \partial_x - \widetilde{r}\partial_x^{-1}\widetilde{q}$$

with

$$\widetilde{q} = -q(\ln q)_{xx} + q^2 r, \quad \widetilde{r} = \frac{1}{q}. \tag{5.4}$$

Taking a product of $\widetilde{r}, \widetilde{q}$ one obtains

$$\widetilde{\rho}_x = -2\widetilde{r}\widetilde{q} = \rho_x + 2(\ln q)_{xx}$$

Thus the DB transformation (5.4) yields

$$\widetilde{\rho} = \rho + 2(\ln q)_x = \rho - 2y - 4x = \rho - 2y_+ - 4x$$
$$= \rho - 2\frac{-1}{2\rho_x}\left(2\left(x\rho_x - \rho\right) + \rho_{xx}\right) - 4x = \rho - 2\frac{\rho}{\rho_x} + \frac{\rho_{xx}}{\rho_x} - 2x \tag{5.5}$$

or

$$\rho^{(\mu,\nu+2)} = \rho^{(\mu,\nu)} - 2\frac{\rho^{(\mu,\nu)}}{\rho_x^{(\mu,\nu)}} + \frac{\rho_{xx}^{(\mu,\nu)}}{\rho_x^{(\mu,\nu)}} - 2x. \tag{5.6}$$

Transformations $\widetilde{\phantom{x}}$ and $\bar{\phantom{x}}$ when acting on variables $J, \bar{J}$ such that:

$$\bar{J} = -rq = \rho_x/2, \quad J = (\ln q)_x = -y - 2x$$

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take the following form [1] :

$$G(J) \equiv J + \left(\ln \left(\bar{J} + J_x\right)\right)_x \qquad\qquad G(\bar{J}) \equiv \bar{J} + J_x \tag{5.7}$$

$$G^{-1}(J) \equiv J - \left(\ln \bar{J}\right)_x \qquad\qquad G^{-1}(\bar{J}) \equiv \bar{J} + \left(\ln \bar{J}\right)_{xx} - J_x , \tag{5.8}$$

where we found it convenient to rewrite actions of $\sim$ and $^-$ as transformations $G$ and $G^{-1}$. It follows from (5.7)-(5.8) that:

$$G(y) = y - \left(\ln \left(y_x + y^2 + 2xy + 2\nu + 4\right)\right)_x , \; G^{-1}(y) = y + \left(\ln \left(y_x - y^2 - 2xy - 2\nu\right)\right)_x . \tag{5.9}$$

The DB transformations $G^{\pm 1}$ are equal to Murata's transformations $T_{\mp}$ first obtained in [11]. This is established after making use of the Painlevé equation (4.3) in (5.9) and identifying parameters $\theta, \alpha$ from [11] with $\mu, -\nu - 1$.

There also exists a set of variables $\jmath, \bar{\jmath}$ related to $J, \bar{J}$ via a Miura transformation [1] such that

$$J = -\jmath - \bar{\jmath} + \frac{\jmath_x}{\jmath} \qquad ; \qquad \bar{J} = \bar{\jmath}\jmath \tag{5.10}$$

In terms of variables $\jmath, \bar{\jmath}$ one can define a discrete symmetry transformation

$$g(\jmath) \equiv \bar{\jmath} - \frac{\jmath_x}{\jmath} \qquad\qquad g\left(\bar{\jmath}\right) \equiv \jmath \tag{5.11}$$

$$g^{-1}(\bar{\jmath}) \equiv \jmath + \frac{\bar{\jmath}_x}{\bar{\jmath}} \qquad\qquad g^{-1}(\jmath) \equiv \bar{\jmath} \tag{5.12}$$

such that [1]

$$g^2 = G , \tag{5.13}$$

when applied on $J, \bar{J}$ defined by equation (5.10).

From the above relations one obtains simple transformation rules:

$$g(y) = y - \left(\ln(-\jmath + y + 2x)\right)_x , \quad g^{-1}(y) = y + \left(\ln(\jmath)\right)_x . \tag{5.14}$$

Acting on solutions $r, q$ or $J, \bar{J}$ of the Painlevé IV equation with $G$ raises $\nu$ by 2 and keeps $\mu$ constant while $g$ when applied on solutions expressed by $\jmath, \bar{\jmath}$ shifts both $\nu$ and $\mu$ by 1 (see below) in such a way that acting twice with $g$ agrees with the formula (5.13). Because of property of the $g$ transformation to shift both parameters of the Painlevé equation this transformation will be very useful in what follows in deriving solutions corresponding to new values of the parameters.

Plugging $r, q$ into expression (5.10) leads to

$$(\ln q)_x = -\bar{\jmath} - \jmath + \jmath_x/\jmath, \qquad rq = -\frac{1}{2}\rho_x = -\bar{\jmath}\jmath , \tag{5.15}$$

which after elimination of $\bar{\jmath}$ yields a Riccati equation for $\jmath$ :

$$rq = \jmath^2 + \jmath(\ln q)_x - \jmath_x . \tag{5.16}$$

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Rewriting equation (4.7) as $2QP + (\mu + \nu) = (2P)^2 + 2P(-Q - 2x) - 2P_x$ and recalling that $Q = -(\ln q)_x - 2x$ and $2QP + (\mu + \nu) = rq$ we find after comparing with relation (5.16) that the solution to this relation is given by

$$j = 2P = -\frac{z}{2y} = \frac{1}{2y}\left(y^2 - y_x + 2xy - 2\mu\right) , \qquad (5.17)$$

where use was made of the first equation in (4.2). Plugging the value of $j$ from (5.17) with two values of $\mu = \pm|\mu|$ into relations (5.14) one obtains two set of transformations $g_\pm : \nu \to \nu + 1$, $\mu^2 \to (|\mu| \pm 1)^2$ and two set of inverse transformations $g_\pm^{-1} : \nu \to \nu - 1$, $\mu^2 \to (|\mu| \pm 1)^2$ recovering expressions found in [8].

As observed in [1], if one sets $S_n = G^n(J)$ and $R_n = G^n(\bar{J})$ then the above set of variables satisfy the equations of motion of the Toda chain:

$$\partial_x S_n = R_{n+1} - R_n, \quad \partial_x R_n = R_n\left(S_n - S_{n-1}\right) .$$

For quantities $\phi_n \equiv \int G^n(J)\, dt$ and $\partial_x \ln \tau_n \equiv \int G^n(\bar{J})\, dt$ one finds on basis of properties (5.7) and (5.8) of the $G$ symmetry :

$$\frac{\tau_{n+1}}{\tau_n} = e^{\phi_n} \qquad ; \qquad e^{\phi_n - \phi_{n-1}} = G^n(\bar{J}) = R_n , \qquad (5.18)$$

from which the Hirota type of Toda chain equation:

$$\partial_x^2 \ln \tau_n = \frac{\tau_{n+1}\, \tau_{n-1}}{\tau_n^2} \qquad (5.19)$$

follows easily. The symmetry structure of $g$ transformations is that of Volterra lattice [1].

## 5.1 Darboux-Bäcklund transformations and "$-2x$ hierarchy"

Things simplify in the case of $q = 0, r = r^{(0)}$ and thus with the initial AKNS Lax operator $L^{(0)} = \partial_x$. A chain of Darboux-Bäcklund transformations define a string of new Lax operators via:

$$L^{(0)} = \partial_x \longrightarrow L^{(k)} = \partial_x + r^{(k)}\partial_x^{-1} q^{(k)}$$

with

$$L^{(k)} = T^{(k)}\left(r^{(k-1)}\right)\partial_x \left(T^{(k)}\left(r^{(k-1)}\right)\right)^{-1}, \quad T^{(k)}\left(r^{(k-1)}\right) = r^{(k-1)}\partial_x \left(r^{(k-1)}\right)^{-1}$$

for $k \geq 1$. The eigenfunction $r^{(0)}$ of $L^{(0)}$ has to satisfy $\partial_{t_n} r^{(0)} = \left(L^{(0)}\right)_+^n r^{(0)} = \partial_x^n r^{(0)}$. The standard choice

$$r^{(0)}(t) = e^{\sum_1^\infty t_n z^n} = \sum_{k=0}^\infty S_k(t) z^k$$

ensures that this is the case. To be able to assign a simple conformal weight to $r$ one can break this sum and just set $r^{(0)}(t) = S_n(t)$ due to the scaling law :

$$S_n(t_\lambda) = \lambda^n S_n(t), \quad t_\lambda = (\lambda x, \lambda^2 t_2, \ldots)$$

which can be expressed as a differential condition (see (2.13)):

$$\left( x\partial_x + 2t_2 \frac{\partial}{\partial t_2} + \ldots \right) S_n(t) = nS_n(t), \quad \rightarrow \quad n = \nu - 1.$$

With this choice the Wronskian expression from [2] becomes :

$$r^{(k)} = r^{(k)} = \frac{W_{k+1}[r^{(0)}, \partial_x r^{(0)}, \ldots, \partial_x^k r^{(0)}]}{W_k[r^{(0)}, \partial_x r^{(0)}, \ldots, \partial_x^{k-1} r^{(0)}]} = \frac{W_{k+1}[S_n, S_{n-1}, \ldots, S_{n-k}]}{W_k[S_n, S_{n-1}, \ldots, S_{n-k+1}]},$$

after using $S_{n-k} = \partial_x^k S_n$.
    Similarly we find for $q^{(k)}$

$$q^{(k)} = -\frac{1}{r^{(k-1)}} = -\frac{W_{k-1}[S_n, S_{n-1}, \ldots, S_{n-k+2}]}{W_k[S_n, S_{n-1}, \ldots, S_{n-k+1}]}, \quad k > 1.$$

We see that the above Darboux-Bäcklund transformation changes the value of $\nu$ by increasing the value of $k$.
    Note that for $t_2 = -1/4$ and $t_3 = t_4 = \ldots = 0$ one gets

$$e^{\sum_1^\infty t_n z^n} = e^{xz - z^2/4} = \sum_{n=0}^\infty \frac{1}{n!} H_n(x) \left( \frac{z}{2} \right)^n$$

and so in this limit $S_n(x) = H_n(x)/2^n n!$. Thus $r^{(0)} = H_n(x)$ and $q^{(0)} = 0$ and $\nu = n + 1$ agrees with what we found above for solution with $q = 0$ and $\nu$ being a positive integer.
    Note that a simple Wronskian formula gives:

$$\begin{aligned} r^{(k)} q^{(k)} &= -\frac{W_{k+1}[S_n, S_{n-1}, \ldots, S_{n-k}] W_{k-1}[S_n, S_{n-1}, \ldots, S_{n-k+2}]}{W_k^2[S_n, S_{n-1}, \ldots, S_{n-k+1}]} \\ &= -\partial_x^2 \ln W_k[S_n, S_{n-1}, \ldots, S_{n-k+1}] \end{aligned}$$

and so one finds

$$\rho^{(k,n)} = 2\partial_x \ln W_k[S_n, S_{n-1}, \ldots, S_{n-k+1}]$$

or in terms of the Hermite polynomials:

$$\rho^{(k,n)} = 2\partial_x \ln W_k[H_n, H_{n-1}, \ldots, H_{n-k+1}], \quad k > 1,$$

which satisfies $\rho$-equation (2.16) with $\nu = n - 2 \cdot k + 1$ and $\mu^2 = (n+1)^2$. That result is in agreement with

$$w^{(k,n)} = \left( \ln r^{(k)} \right)_x - 2x = \partial_x \left( \ln \frac{W_{k+1}[H_n, H_{n-1}, \ldots, H_{n-k}]}{W_k[H_n, H_{n-1} \ldots, H_{n-k+1}]} \right) - 2x$$

satisfying (4.4) with the same values of parameters.

There exists another chain of successive DB transformations, which also builds the AKNS Lax pseudo-differential operator starting from the "pure" derivative $\partial_x$ by :

$$L^{(0)} = \partial_x \longrightarrow L^{(k)} = \partial_x + r^{(k)} \partial_x^{-1} q^{(k)}$$

with

$$L^{(k)} = T^{(k)}\left(q^{(k-1)}\right) \partial_x \left(T^{(k)}\left(q^{(k-1)}\right)\right)^{-1}, \quad T^{(k)}\left(q^{(k-1)}\right) = \left(q^{(k-1)}\right)^{-1} \partial_x^{-1} q^{(k-1)}$$

for $k \geq 1$. The adjoint eigenfunction $q^{(0)}$ of $L^{(0)}$ has to satisfy $\partial_{t_n} q^{(0)} = -\left(L^{(0)}\right)_+^{*n} q^{(0)} = (-1)^{n+1} \partial_x^n q^{(0)}$. The standard choice

$$q^{(0)}(t) = e^{\sum_1^\infty (-1)^{n+1} t_n z^n} = \sum_{k=0}^\infty \widehat{S}_k(t) z^k$$

ensures that this is the case. To be able to assign a simple conformal weight for $q$ it is convenient to break this sum by setting $q^{(0)}(t) = \widehat{S}_n(t)$ due to the scaling law :

$$\widehat{S}_n(t_\lambda) = \lambda^n \widehat{S}_n(t), \quad t_\lambda = (\lambda x, \lambda^2 t_2, \ldots)$$

With this choice the Wronskian expression for $q^{(k)}$ becomes

$$q^{(k)} = -\frac{W_{k+1}[q^{(0)}, \partial_x q^{(0)}, \ldots, \partial_x^k q^{(0)}]}{W_k[q^{(0)}, \partial_x q^{(0)}, \ldots, \partial_x^{k-1} q^{(0)}]} = -\frac{W_{k+1}[\widehat{S}_n, \widehat{S}_{n-1}, \ldots, \widehat{S}_{n-k}]}{W_k[\widehat{S}_n, \widehat{S}_{n-1}, \ldots, \widehat{S}_{n-k+1}]}$$

after using $\widehat{S}_{n-k} = \partial_x^k \widehat{S}_n$. Similarly we find for $r^{(k)}$

$$r^{(k)} = r^{(k)} = \frac{1}{q^{(k-1)}} = -\frac{W_{k-1}[\widehat{S}_n, \widehat{S}_{n-1}, \ldots, \widehat{S}_{n-k+2}]}{W_k[\widehat{S}_n, \widehat{S}_{n-1}, \ldots, \widehat{S}_{n-k+1}]}, \quad k > 1.$$

Note that for $t_2 = -1/4$ and $t_3 = t_4 = \ldots = 0$ one gets

$$e^{\sum_1^\infty t_n (-1)^{n+1} z^n} = e^{xz + z^2/4} = \sum_{n=0}^\infty \frac{1}{n!} \widehat{H}_n(x) \left(\frac{z}{2}\right)^n$$

and so in this limit $\widehat{S}_n(x) = \widehat{H}_n(x)/2^n n!$, where $\widehat{H}_n(x)$ is introduced in eq. (3.4). Thus $q^{(0)} = \widehat{H}_n(x)$ and $r^{(0)} = 0$ and $\nu = -n - 1$ agrees with what we found above for solution with $r = 0$ and $\nu < 0$ and integer. Indeed

$$\widehat{\rho}^{(k,n)} = 2\partial_x \ln W_k[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+1}]$$

satisfies $\rho$-equation (2.16) with $\nu = -n + 2 \cdot k - 1$ and $\mu^2 = (n+1)^2$.

To summarize we found the solutions to the $\rho$-equation (2.16) with $\nu, \mu$ parameters given by :

$$\rho^{(k,n)} = 2\partial_x \ln W_k[H_n, H_{n-1}, \ldots, H_{n-k+1}], \quad \nu = n - 2 \cdot k + 1, \ \mu^2 = (n+1)^2$$
$$\widehat{\rho}^{(k,n)} = 2\partial_x \ln W_k[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+1}], \quad \nu = -n + 2 \cdot k - 1, \ \mu^2 = (n+1)^2$$

$$(5.20)$$

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with $n, k = 1, 2, 3, \ldots$. Corresponding solutions to the Painlevé eqs. (4.3) and (4.4), referred to as "$-2x$-hierarchy"[5, 6] are, respectively,

$$w^{(k,n)} = \partial_x \ln r^{(k-1)} - 2x = \partial_x \ln \frac{W_k[H_n, H_{n-1}, \ldots, H_{n-k+1}]}{W_{k-1}[H_n, H_{n-1}, \ldots, H_{n-k+2}]} - 2x \tag{5.21}$$

$$\nu = n - 2 \cdot k + 1, \quad \mu^2 = (n+1)^2$$

$$y^{(k,n)} = -\partial_x \ln q^{(k-1)} - 2x = -\partial_x \ln \frac{W_k[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+1}]}{W_{k-1}[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+2}]} - 2x \tag{5.22}$$

$$\nu = -n + 2 \cdot k - 1, \quad \mu^2 = (n+1)^2$$

with $n, k = 1, 2, 3, \ldots$.

## 5.2 Darboux-Bäcklund transformations and "$-1/x$-hierarchy"

We will now examine an hierarchy of solutions which can be obtained by the Darboux-Bäcklund approach from remaining basic solutions listed in eqs. (3.6) and (3.5). Namely, we consider as starting points $q(x) = \exp\left(-x^2\right) H_\nu(x)$ with $\nu$ being a positive integer and $r(x) = e^{x^2} \widehat{H}_{-\nu}(x)$ for $\nu \leq 0$ and integer.

We begin with $q^{(0)} = \exp\left(-x^2\right) H_m(x)$, $r^{(0)} = 0$ and corresponding

$$y = -\partial_x \ln \left(e^{-x^2} H_m\right) - 2x = -\partial_x \ln H_m,$$

which satisfies eq. (4.3) with $\nu = m$ and $\mu^2 = m^2$.

The iteration procedure (5.7) expressed here in terms of the AKNS variables :

$$q^{(1)} = G(q^{(0)}) = -q^{(0)} \left(\ln q^{(0)}\right)_{xx} + (q^{(0)})^2 r^{(0)},$$

$$q^{(k+1)} = G(q^{(k)}) = -q^{(k)} \left(\ln q^{(k)}\right)_{xx} + \left(q^{(k)}\right)^2 \frac{1}{q^{(k-1)}}, \quad k > 0 \tag{5.23}$$

with $q^{(0)}, r^{(0)}$ as defined above is solved by

$$q^{(k)} = \frac{W_{k+1}[e^{-x^2} H_m, e^{-x^2} H_{m+1}, \ldots, e^{-x^2} H_{m+k}]}{W_k[e^{-x^2} H_m, e^{-x^2} H_{m+1}, \ldots, e^{-x^2} H_{m+k-1}]}. \tag{5.24}$$

Using the recurrence relation $(\exp(-x^2)H_m)_x = -\exp(-x^2)H_{m+1}$ for Hermite's polynomials allows us to cast the above expression for $q^{(k)}$ into a simple form

$$q^{(k)} = (-1)^k \frac{W_{k+1}}{W_k}$$

where

$$W_{k+1} = W_{k+1}[f, f_x, f_{xx}, \ldots, f_{\underbrace{x \cdots x}_{k}}], \quad f = e^{-x^2} H_m$$

Then applying the $G$ transformation yields:

$$\begin{aligned}
G(q^{(k)}) &= (-1)^k \frac{W_k}{W_{k+1}} \left[ \frac{1}{W_k^2} \left( -W_{k+1} W_{k+1}'' + (W_{k+1}')^2 \right) + \frac{W_{k+1}^2 W_k''}{W_k^3} \right. \\
&\qquad \left. - \frac{W_{k+1}^2 (W_k')^2}{W_k^4} - \frac{W_{k+1}^3 W_{k-1}}{W_k^4} \right] \\
&= (-1)^{k+1} \frac{W_k^2 W_{k+2}}{W_{k+1} W_k^2} = (-1)^{k+1} \frac{W_{k+2}}{W_{k+1}}
\end{aligned}$$

where use was repeatedly made of the identity $W_k W_k'' - (W_k')^2 = W_{k-1} W_{k+1}$ for an arbitrary argument $f$. Thus, $G(q^{(k)}) = q^{(k+1)}$ and (5.24) is established by an induction argument.

Taking into account the identity

$$W_k[e^{-x^2} P_1, \ldots, e^{-x^2} P_k] = W_k[P_1, \ldots, P_k] \, e^{-k\,x^2}$$

and setting $m = n - k + 1$ we obtain the following solution to the Painlevé eq. (4.3):

$$\begin{aligned}
y^{(k,n)} &= -\partial_x \ln \frac{W_{k+1}[e^{-x^2} H_{n+1}, e^{-x^2} H_n, \ldots, e^{-x^2} H_{n-k+1}]}{W_k[e^{-x^2} H_n, e^{-x^2} H_{n-1}, \ldots, e^{-x^2} H_{n-k+1}]} - 2x \\
&= \partial_x \ln \frac{W_k[H_n, H_{n-1}, \ldots, H_{n-k+1}]}{W_{k+1}[H_{n+1}, H_n, \ldots, H_{n-k+1}]}
\end{aligned} \qquad (5.25)$$

with $\nu = n + k + 1$ and $\mu^2 = (n - k + 1)^2$. Note that $y^{(0,n)}$ reproduces solution $-\partial_x \ln H_n$ encountered above.

We now turn our attention to $r(x) = r^{(0)}(x) = e^{x^2} \widehat{H}_m(x)$ and corresponding $w = \partial_x \ln \left( \widehat{H}_m \right)$, which satisfies (4.4) with $\nu = -m$ and $\mu^2 = m^2$ in agreement with eq. (3.5). Repeating steps applied above to find $q^{(k)}$ one easily establishes that

$$r^{(-k)} = \frac{W_{k+1}[e^{x^2} \widehat{H}_m, e^{x^2} \widehat{H}_{m+1}, \ldots, e^{x^2} \widehat{H}_{m+k}]}{W_k[e^{x^2} \widehat{H}_m, e^{x^2} \widehat{H}_{m+1}, \ldots, e^{x^2} \widehat{H}_{m+k-1}]} . \qquad (5.26)$$

happens to be a solution to the iteration relations:

$$\begin{aligned}
r^{(-1)} &= G^{-1}(r^{(0)}) = r^{(0)} \left( \ln r^{(0)} \right)_{xx} - (r^{(0)})^2 q^{(0)}, \\
r^{(-k-1)} &= G^{-1}(r^{(-k)}) = r^{(-k)} \left( \ln r^{(-k)} \right)_{xx} + \left( r^{(-k)} \right)^2 \frac{1}{r^{(-k+1)}}, \quad k > 0 .
\end{aligned} \qquad (5.27)$$

with $q^{(0)} = 0$. Plugging expression (5.26) into expression for $w$ from relation (4.1) and setting $m = n - k + 1$ one obtains solutions

$$w^{(k,n)} = -\partial_x \ln \frac{W_k[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+1}]}{W_{k+1}[\widehat{H}_{n+1}, \widehat{H}_n, \ldots, \widehat{H}_{n-k+1}]} = \partial_x \ln \frac{W_{n-k+1}[H_{n+1}, H_n, \ldots, H_{k+1}]}{W_{n-k+1}[H_n, H_{n-1}, \ldots, H_k]} \qquad (5.28)$$

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to the Painlevé eq. (4.4) with $\nu = -(n+k+1)$, $\mu = (n-k+1)^2$. In obtaining (5.28) use was made of a Wronskian identity,

$$\frac{W_{k+1}[\widehat{H}_{n+1}, \widehat{H}_n, \ldots, \widehat{H}_{n-k+1}]}{W_k[\widehat{H}_n, \widehat{H}_{n-1}, \ldots, \widehat{H}_{n-k+1}]} = C_{n,k} \frac{W_{n-k+1}[H_{n+1}, H_n, \ldots, H_{k+1}]}{W_{n-k+1}[H_n, H_{n-1}, \ldots, H_k]} ,$$

where $C_{n,k}$ are some combinatorial constants.

Expressions (5.25) and (5.28) define "$-1/x$-hierarchy" [5].

## 5.3 Darboux-Bäcklund transformations and "$-2x/3$-hierarchy"

### 5.3.1 Solutions with $\mu^2 = (1/3)^2, (2/3)^2$

Consider now solution (3.2) of the $\rho$-equation (2.16) with $\mu^2 = (2/3)^2, \nu = 0$. According to relation (4.11) the corresponding two solutions of equations (4.3), (4.4) are :

$$y_+ = -\frac{1}{x}\left(\frac{2}{3}x^2 + 1\right) = -(\ln q)_x - 2x, \quad y_- = -\frac{1}{x}\left(\frac{2}{3}x^2 - 1\right) = (\ln r)_x - 2x,$$

with the following solutions of the string equations (2.14) for $\nu = 0$:

$$q = q^{(0)} = -\frac{2}{3}xe^{-2x^2/3} = F_0^{(1)}e^{-2x^2/3}, \quad r = r^{(0)} = \frac{2}{3}xe^{2x^2/3} = \widehat{F}_0^{(1)}e^{2x^2/3} . \quad (5.29)$$

In the above equation we employed notation involving polynomials :

$$F_n^{(k)} = \frac{e^{x^2/3}}{2^n n!}\frac{d^{3n+k}}{dx^{3n+k}}e^{-x^2/3}, \quad \widehat{F}_n^{(k)} = \frac{e^{-x^2/3}}{2^n n!}\frac{d^{3n+k}}{dx^{3n+k}}e^{x^2/3}, \quad (5.30)$$

defined for $k = 0, 1, 2$, $n = 0, 1, 2, \ldots$. By simple rescaling of their arguments polynomials $F_n^{(k)}, \widehat{F}_n^{(k)}$ become proportional to Hermite polynomials $H_m, \widehat{H}_m$ for certain values of $m$.

Acting on the initial configuration (5.29) successively with the DB transformations as in (5.23) one arrives at the following Wronskian representation in terms of the ratios of Wronskians :

$$q^{(n)} = \frac{W_{n+1}[F_0^{(1)}, F_1^{(1)}, \ldots, F_n^{(1)}]}{W_n[F_0^{(1)}, F_1^{(1)}, \ldots, F_{n-1}^{(1)}]}\exp\left(-\frac{2x^2}{3}\right), \quad r^{(n)} = 1/q^{(n-1)} \quad (5.31\text{a})$$

$$\mu = \pm\frac{2}{3}, \quad \nu = 2n, \quad n \geq 0 .$$

Acting with negative powers of $G$ as in (5.27) yields :

$$r^{(-n)} = \frac{W_{n+1}[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_n^{(1)}]}{W_n[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_{n-1}^{(1)}]}\exp\left(\frac{2x^2}{3}\right), \quad q^{(-n)} = -1/r^{(-n+1)} \quad (5.31\text{b})$$

$$\mu = \pm\frac{2}{3}, \quad \nu = -2n, \quad n \geq 0 .$$

The first few solutions are

$$q^{(1)} = \frac{2\left(-12x^2 + 4x^4 - 9\right)}{27x}e^{-\frac{2x^2}{3}}, \qquad q^{(2)} = -\frac{8x(504x^4 - 192x^6 + 16x^8 - 2835)}{243(-12x^2 + 4x^4 - 9)}e^{-\frac{2x^2}{3}}$$

$$r^{(-1)} = \frac{2\left(12\,x^2 - 9 + 4\,x^4\right)}{27x}e^{\frac{2x^2}{3}}, \qquad r^{(-2)} = \frac{8x\left(-2835 + 504\,x^4 + 192\,x^6 + 16\,x^8\right)}{243(12\,x^2 - 9 + 4\,x^4)}e^{\frac{2x^2}{3}}.$$

Applying recursively (5.3) one gets closed Wronskian expressions for the $\rho$-function:

$$\rho^{(n)} = G^n(\rho) = \rho - 2n\frac{4x}{3} + 2\left(\ln W_n[F_0^{(1)}, F_1^{(1)}, \ldots, F_{n-1}^{(1)}]\right)_x,$$

$$\rho^{(-n)} = G^{-n}(\rho) = \rho + 2n\frac{4x}{3} + 2\left(\ln W_n[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_{n-1}^{(1)}]\right)_x,$$

which are solutions of the $\rho$-equation (2.16) with $\nu = \pm 2n, n = 1, 2, 3, \ldots$, respectively. In particular

$$\rho^{(\pm 1)} = \frac{2}{27x}\left(4x^4 + 27 \pm 36x^2\right)$$

satisfies the $\rho$-equation (2.16) with $\mu^2 = \frac{4}{9}, \nu = \pm 2$.

Consider now the basic solutions (3.1) of $\rho$-equation (2.16). Plugging (3.1) (with a plus sign) into relation (4.11) one obtains the following two solutions of equation (4.3):

$$y = y_+ = -\frac{2}{3}x, \qquad\qquad \mu = \pm\frac{1}{3}, \quad \nu = -1 \qquad (5.32)$$

$$w = y_- = -\frac{2}{3}x\frac{\frac{16}{9}x^2 - \frac{8}{3}}{\frac{16}{9}x^2 + \frac{8}{3}}, \qquad \mu = \pm\frac{1}{3}, \quad \nu = -1 \qquad (5.33)$$

Eq. (5.33) corresponds to the following AKNS variables:

$$r = r^{(0)} = \frac{2}{3}\left(\frac{2}{3}x^2 + 1\right)e^{2x^2/3} = \widehat{F}_0^{(2)}e^{2x^2/3},$$

$$q = q^{(0)} = -e^{-2x^2/3} = -F_0^{(0)}e^{-2x^2/3}, \qquad\qquad (5.34)$$

Applying the DB transformations $G, G^{-1}$ generalizes the above basic solutions to

$$q^{(n)} = -\frac{W_{n+1}[F_0^{(0)}, F_1^{(0)}, \ldots, F_n^{(0)}]}{W_n[F_0^{(0)}, F_1^{(0)}, \ldots, F_{n-1}^{(0)}]}\exp\left(-\frac{2x^2}{3}\right), \qquad r^{(n+1)} = 1/q^{(n)} \qquad (5.35a)$$

$$\mu = \pm\frac{1}{3}, \quad \nu = -1 + 2n, \quad n \geq 0$$

and

$$r^{(-n)} = \frac{W_{n+1}[\widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_n^{(2)}]}{W_n[\widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_{n-1}^{(2)}]}\exp\left(\frac{2x^2}{3}\right), \qquad q^{(-n)} = -1/r^{(-n+1)} \qquad (5.35b)$$

$$\mu = \pm\frac{1}{3}, \quad \nu = -1 - 2n, \quad n \geq 0.$$

Moreover one obtains the following Wronskian expressions for the chain of associated solutions to the $\rho$-equation:

$$\rho^{(n)} = G^n(\rho) = \rho - 2n\frac{4x}{3} + 2\left(\ln W_n[F_0^{(0)}, F_1^{(0)}, \ldots, F_{n-1}^{(0)}]\right)_x, \quad \nu = 1 + 2n,$$

$$\rho^{(-n)} = G^{-n}(\rho) = \rho + 2n\frac{4x}{3} + 2\left(\ln W_n[\widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_{n-1}^{(2)}]\right)_x, \quad \nu = 1 - 2n$$

with $n = 1, 2, 3, \ldots$.

Plugging (3.1) (with a minus sign) into relation (4.11) one obtains the following two solutions of equation (4.3):

$$y = y_+ = -\frac{2}{3}x\frac{3 + 2x^2}{-3 + 2x^2}, \qquad\qquad \mu = \pm\frac{1}{3}, \ \nu = +1 \qquad (5.36)$$

$$w = y_- = -\frac{2}{3}x, \qquad\qquad \mu = \pm\frac{1}{3}, \ \nu = +1 \qquad (5.37)$$

which corresponds to the following AKNS variables:

$$r = r^{(0)} = \widehat{F}_0^{(0)} e^{2x^2/3}, \quad q = q^{(0)} = -F_0^{(2)} e^{-2x^2/3}, \quad \mu = \pm\frac{1}{3}, \nu = 1.$$

Applying the DB transformations $G, G^{-1}$ generalizes the above basic solutions to

$$q^{(n)} = \frac{W_{n+1}[F_0^{(2)}, F_1^{(2)}, \ldots, F_n^{(2)}]}{W_n[F_0^{(2)}, F_1^{(2)}, \ldots, F_{n-1}^{(2)}]} \exp\left(-\frac{2x^2}{3}\right), \quad r^{(n)} = 1/q^{(n-1)} \qquad (5.38a)$$

$$\mu = \pm\frac{1}{3}, \quad \nu = 1 + 2n, \ n \geq 0$$

and

$$r^{(-n)} = \frac{W_{n+1}[\widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_n^{(0)}]}{W_n[\widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_{n-1}^{(0)}]} \exp\left(\frac{2x^2}{3}\right), \quad q^{(-n)} = -1/r^{(-n+1)} \qquad (5.38b)$$

$$\mu = \pm\frac{1}{3}, \quad \nu = 1 - 2n, \ n \geq 0,$$

The Wronskian identities :

$$W_{n+1}[F_0^{(0)}, F_1^{(0)}, \ldots, F_n^{(0)}] = -W_n[F_0^{(2)}, F_1^{(2)}, \ldots, F_{n-1}^{(2)}],$$

$$W_{n+1}[\widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_n^{(0)}] = W_n[\widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_{n-1}^{(2)}]$$

ensure that the solutions $y, w$ of the Painlevé IV equations generated by expressions in eqs. (5.38) coincide with those solutions $y, w$ which originate from equations (5.35) for equal parameter $\nu$ (accomplished by shifting $n$ to $n \pm 1$ when going from (5.35) to (5.38)).

Moreover one obtains the following Wronskian expressions for the $\rho$-function:

$$\rho^{(-n)} = G^{-n}(\rho) = \rho + 2n\frac{4x}{3} + 2\left(\ln W_n[\widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_{n-1}^{(0)}]\right)_x, \quad n = 1, 2, 3, \ldots$$

$$\rho^{(n)} = G^n(\rho) = \rho - 2n\frac{4x}{3} + 2\left(\ln W_n[F_0^{(2)}, F_1^{(2)}, \ldots, F_{n-1}^{(2)}]\right)_x, \quad n = 1, 2, 3, \ldots$$

which satisfies the $\rho$-equation (2.16) with $\mu^2 = \frac{4}{9}$, $\nu = 1 \pm 2n$.

Wronskians $W_n[F_0^{(1)}, F_1^{(1)}, \ldots, F_{n-1}^{(1)}]$ and $W_n[\widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_{n-1}^{(0)}]$ are proportional to the Okamoto polynomials $R_n, Q_n$ as defined in [5].

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### 5.3.2 Solutions with $\mu = \pm 4/3, \pm 5/3, \ldots$

To reach expressions for solutions with $\mu = \pm 4/3, \pm 5/3, \ldots$ we have employed transformations $g, g^{-1}$ with their properties of raising and lowering $\mu$ and $\nu$ by one when applied to $\jmath, \bar{\jmath}$ configuration. This property of the $g, g^{-1}$ transformations ensures that we reach all allowed values of the Painlevé IV parameters following the zig-zag DB orbits through the $(\mu, \nu)$-plane.

We now present results obtained by applying $g, g^{-1}$ transformations to the three basic cases with the values $\mu^2 = (2/3)^2, (1/3)^2$ presented in equations (5.31), (5.35) and (5.38) in the previous subsection.

We start with a chain of solutions (5.31) to the Painlevé IV equations with $\mu^2 = (2/3)^2$. Through the successive actions of $g, g^{-1}$ transformations these solutions generalize to :

$$y = -\left( \ln \left( \frac{W_{k+m+2}\left[F_0^{(1)}, F_1^{(1)}, \ldots, F_k^{(1)}, F_0^{(2)}, F_1^{(2)}, \ldots, F_m^{(2)}\right]}{W_{k+m+1}\left[F_0^{(1)}, F_1^{(1)}, \ldots, F_{k-1}^{(1)}, F_0^{(2)}, F_1^{(2)}, \ldots, F_m^{(2)}\right]} \right) \right)_x - \frac{2x}{3}, \quad (5.39a)$$

$$\mu^2 = \left(\frac{2}{3} + m + 1\right)^2, \qquad \nu = 2k - m - 1$$

$$w = \left( \ln \left( \frac{W_{k+m+2}\left[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_k^{(1)}, \widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_m^{(2)}\right]}{W_{k+m+1}\left[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_{k-1}^{(1)}, \widehat{F}_0^{(2)}, \widehat{F}_1^{(2)}, \ldots, \widehat{F}_m^{(2)}\right]} \right) \right)_x - \frac{2x}{3}, \quad (5.39b)$$

$$\mu^2 = \left(\frac{2}{3} + m + 1\right)^2, \qquad \nu = 1 + m - 2k$$

and solve the Painlevé IV equations (4.3) and (4.4), respectively, for the positive integers $k, m \geq 0$.

Generalizing through the DB approach the system of solutions given in eqs. (5.35) leads to :

$$y = -\left( \ln \left( \frac{W_{k+m+2}\left[F_0^{(1)}, F_1^{(1)}, \ldots, F_m^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_k^{(0)}\right]}{W_{k+m+1}\left[F_0^{(1)}, F_1^{(1)}, \ldots, F_m^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_{k-1}^{(0)}\right]} \right) \right)_x - \frac{2x}{3}, \quad (5.40a)$$

$$\mu^2 = \left(-\frac{1}{3} + m + 1\right)^2 = \left(\frac{2}{3} + m\right)^2, \qquad \nu = 2k - m - 2$$

$$w = \left( \ln \left( \frac{W_{k+m+2}\left[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_m^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_k^{(0)}\right]}{W_{k+m+1}\left[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_m^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_{k-1}^{(0)}\right]} \right) \right)_x - \frac{2x}{3}, \quad (5.40b)$$

$$\mu^2 = \left(-\frac{1}{3} + m + 1\right)^2 = \left(\frac{2}{3} + m\right)^2, \qquad \nu = -2k + m + 2$$

written in terms of two positive integers $k$ and $m$.

By applying the same procedure to solutions given in equations (5.38) one obtains

a new class of solutions :

$$y = - \left( \ln \left( \frac{W_{k+m+2}\big[F_0^{(1)}, F_1^{(1)}, \ldots, F_k^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_m^{(0)}\big]}{W_{k+m+1}\big[F_0^{(1)}, F_1^{(1)}, \ldots, F_{k-1}^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_m^{(0)}\big]} \right) \right)_x - \frac{2x}{3}, \quad \text{(5.41a)}$$

$$\mu^2 = \left( \frac{1}{3} + m \right)^2, \qquad \nu = 2k - m - 1$$

$$w = \left( \ln \left( \frac{W_{k+m+2}\big[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_k^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_m^{(0)}\big]}{W_{k+m+1}\big[\widehat{F}_0^{(1)}, \widehat{F}_1^{(1)}, \ldots, \widehat{F}_{k-1}^{(1)}, \widehat{F}_0^{(0)}, \widehat{F}_1^{(0)}, \ldots, \widehat{F}_m^{(0)}\big]} \right) \right)_x - \frac{2x}{3}, \quad \text{(5.41b)}$$

$$\mu^2 = \left( \frac{1}{3} + m \right)^2, \qquad \nu = m - 2k + 1$$

written in terms of two positive integers $k, m \geq 0$.

It is important to point out that the first two of the above three classes of solutions overlap in their values of the $\mu$ and $\nu$ parameters for $\mu^2 > (2/3)^2$ and in such cases the corresponding solutions coincide. For instance, solutions (5.39a) with $\mu^2 = (2/3 + m + 1)^2$, $\nu = 2k - m - 1$ are equal to solutions (5.40a) with $\mu^2 = (-1/3 + m' + 1)^2$, $\nu = 2k' - m' - 2$ when $m' = m + 1$ and $k' = k + 1$. Thus, for any solution $y_{m,k}$ or $w_{m,k}$ from (5.39) there exists an identical solution $y_{m'=m+1,k'=k+1}$ or $w_{m'=m+1,k'=k+1}$ from (5.40). As encountered before in the text, an equality between two classes of Painlevé IV solutions derived by action of the DB transformations stems from existence of a special Wronskian identity. In this case the relevant identity is given by :

$$W_{k+m+2}\big[F_0^{(1)}, F_1^{(1)}, \ldots, F_k^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_m^{(0)}\big] =$$
$$= K_{k,m} W_{k+m+4}\big[F_0^{(1)}, F_1^{(1)}, \ldots, F_{m+1}^{(1)}, F_0^{(0)}, F_1^{(0)}, \ldots, F_{k+1}^{(0)}\big],$$

with certain combinatorial constants $K_{k,m}$.

# 6 Conclusions and Outlook

We presented here a systematic and self-contained derivation of rational solutions to the Painlevé IV equation using the method of the Darboux-Bäcklund transformations of the particular reduction of the AKNS pseudo-differential Lax hierarchy. By studying the orbits of the Darboux-Bäcklund transformations originating from few seeds solutions we were able to find closed expressions for solutions associated to all allowed values of the Painlevé IV parameters. The explicit expressions for all Wronskian representations derived here seem to be in general agreement with determinant formulas obtained earlier in [10, 12, 6] by alternate methods.

In a separate publication we will discuss the Darboux-Bäcklund transformations versus the (affine) Weyl group symmetry of the Painlevé IV equation [13].

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